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The effect of slip boundary condition on the flow of granular materials: a continuum approach

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Abstract

In the previous studies we have looked at a fully developed flow under the assumption that the granular materials adhere to the boundary. Whether one uses the continuum approach or the kinetic theory approach in modeling of the granular materials, slip may occur at the wall, especially when the interstitial fluid is a gas, and therefore the classical assumption of adherence boundary condition at the wall no longer applies. The steady, fully developed flow of granular materials down an inclined plane subject to slip at the wall is studied numerically. This is a non-linear boundary value problem. The results for the velocity profiles are presented in terms of appropriate dimensionless numbers. Published by Elsevier Science Ltd.

Keywords: Granular materials; Slip boundary condition; Continuum approach; Inclined flow

1. Introduction

Granular materials present one of the most challenging areas of research in mechanics. Experimental studies in soil mechanics dominated this field for a long time. Until a few decades ago where theoretical studies based on modern continuum mechanics started to be used, there were very few theories, based on the foundation of mechanics. In the late 1970s, many researches observed the similarities between the flow (or the behavior) of solid particles and that of gas molecules. Thus, a whole new field of kinetic theory of granular materials was initiated. At the same time, with the advances in numerical techniques and computation, particle simulation became feasible and meaningful. Thus, at the present time, there are a few ways of looking at the complex behavior of granular materials. These different ways, i.e., experiment, kinetic theory, particle simulation, continuum theory, etc., should not be looked upon as contradictory to each other; nor should any one of these methods be taken as the only way, the exclusive way, to understand granular materials. Most of the time, these methods are complementary to each other, and at times some may be more appropriate for a given case, i.e., flow regime, etc.

In recent years there has been considerable interest in understanding the behavior of granular materials because of the relevance that they have to several technological problems. This includes the handling of such substances as coal, agricultural products, fossil-fuel energy, metal ores, crushed oil

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Nomenclature

C_f, C_g	the sliding coefficients
$\mathbf{D}$	symmetric part of the velocity gradient
e	the coefficient of restitution
f	defined by Eq. (21)
f^*	dimensionless slip factor
h	boundary thickness
$\mathbf{n}$	a unit normal vector
N	average value for the volume fraction condition, Eq. (37)
Q	average volume fraction, Eq. (30)
R_1, R_2	dimensionless numbers (Eq. (42))
R_3, R_{21}	dimensionless numbers (Eq. (43))
$\mathbf{T}$	the Cauchy stress tensor
$\mathbf{u}$	the velocity of the particles
u_s	the slip velocity
u^*	dimensionless velocity
U_0	a reference velocity
u_s^*	dimensionless slip velocity
y	y-coordinate

Greek letters

α	the angle of inclination of the plane
β_0	similar to pressure in a compressible fluid and is given by an equation of state
β_2	akin to the second coefficient of viscosity in a compressible fluid
β_1,β_4	the material parameters that reflect the distribution of the granular materials
β_3	the viscosity of the granular materials,
β_{30}	a constant
ς	a constant to describe slip, Eq. (11)
θ	the fluctuating energy of the flow
κ	defined by Eq. (16)
ν	the volume fraction of the solid
ξ	dimensionless distance
ρ_s	the actual density of the grains
ρ^*	defined by Eq. (19)
σ	the particle diameter
τ_w	the wall shear stress
τ	defined by Eq. (15)
χ	defined by Eq. (17)
Φ_0	defined by Eq. (18)

shale, dry chemicals, rocket propellants, fertilizers, cement, sand and other particulate solids. Furthermore, the process of fluidization of coal particles and its effect on combustion, the mechanics of avalanches and other natural disaster that involve the flow of powders and bulk solids present challenging problems. In addition, flowing granular streams are being considered for some advanced concepts for solar power plants and fusion reactor chambers. Many situations, such as discharge through bin outlets, flow through hoppers and chutes, pneumatic transport of coals, fluidized beds, etc., require information on particle properties, flow patterns, concentration profiles, etc. [1,2].

Since granular materials conform to the shape of the vessel containing them, they can be considered fluid-like. However, unlike fluids, they can also be heaped. Characterizing bulk solids is difficult mainly because small variations in some of the primary properties such as size, shape, hardness, particle density, and surface roughness can result in very different behavior. Furthermore, secondary factors (such as the presence or the absence of moisture, and ambient temperature) that are not directly associated with the particles, but are associated with the environment can have a significant effect on the behavior of the bulk solids. The concept of "granular materials" covers the combined range of granular powders and granular solids with components ranging in size from about 10 μm to 3 mm. A powder is composed of particles up to 100 μm with further subdivision into ultrafine (0.1–1.0 μm), superfine (1–10 μm), or granular (10–100 μm) particles. A granular solid consists of materials ranging from about 100 to 3000 μm [3].

Due to their complexity, modeling granular materials would require a fusion of ideas from the mechanics of fluids and solids. For example, granular materials exhibit phenomena such as yield stress and normal stress differences in simple shear flow (a phenomenon usually referred to in the field as dilatancy [4]) characteristic of materials which require non-linear constitutive modeling. The central role played by this phenomenon in determining the characteristics of sand and other granular materials was recognized early in the development of the theories for modeling granular materials. Interestingly, a constitutive model that was proposed for wet sand [5], enjoyed a good bit of popularity as a model for non-Newtonian fluids before losing its appeal. Thus, modeling granular materials and slurries can be very complex and must draw on our experiences from non-linear fluid and non-linear solid theories. One approach in the modeling of granular materials is to treat it as a *continuum*, which assumes that the material properties of the ensemble may be represented by continuous functions so that the medium may be divided infinitely without losing any of its defining properties. One of the early continuum models for flowing granular materials based on the principles of modern continuum mechanics was proposed by Goodman and Cowin [6,7]. This work was subsequently modified and improved upon by other investigators [1,8–14]. Another approach used in the modeling of granular materials is based on techniques used in the *kinetic theory*. Such an approach is appropriate for modeling dilute and rapidly flowing granular materials where collisions dominate. A detailed discussion of such modeling can be found for example, in the articles of Jenkins and Savage [15], Lun et al. [16], Boyle and Massoudi [17]. There are quite a few excellent review articles which discuss many of the relevant issues to the flow of granular materials. We refer the reader to Savage [2], Hutter and Rajagopal [18], Mehta [19], and de Gennes [20].

There are several interesting problems which not only present interesting cases from a mathematical point of view (such as existence and uniqueness of the solution, cf. [21]), but also from an applied and practical point of view. These problems include gravitational flows such as flow in vertical channels (cf. [6,22,23]), inclined flows (cf. [10] and many others), chute flows [24,25], bins and hoppers (cf. [26–29,51] among others), etc.

In the present work we study the flow of granular materials down an inclined plane. In a previous study [23], we looked at a fully developed flow over a heated inclined plane under the assumption that the granular materials adhere to the boundary. This work investigates the problem allowing for the possibility of slip the surface. This problem, i.e., flow down an inclined plane, has received special attention due to the simplicity of the equations. A recent review by Anderson and Jackson [30] looks at a few of such solutions.

In Section 2, we give a brief outline of the constitutive equation which we have been using in the previous studies. The details of this formulation are given in the works of Rajagopal and Massoudi. In Section 3 which is the main emphasis in this paper, we provide a basic review of the slip boundary condition in continuum mechanics, and propose a slip boundary condition for the present study, based on the ideas in classical fluid dynamics and that of Hutter, specifically for granular materials. In Section 4, we give a summary of the numerical scheme (the details can be found in Ref. [31]), and representative profiles for velocity as a function of various dimensionless numbers and slip factor are given.

2. Governing equations

We will now derive equations governing the flow of granular materials down an inclined plane. The basic equations are the conservation of mass

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1}$$

where $\partial/\partial t$ is the partial derivative with respect to time, and the balance of linear momentum which is

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \operatorname{div}\, \mathbf{T} + \rho \mathbf{b}, \tag{2}$$

where $\mathrm{d}/\mathrm{d}t$ is the material time derivative and $\mathbf{b}$ is the body force vector. Since we are interested in the isothermal flow with no chemical reactions, etc., the conservation of energy and the entropy inequality are not used in this analysis. The Cauchy stress $\mathbf{T}$ for a granular material is given by [1,6,7,52]

$$\mathbf{T} = \{\beta_0(v) + \beta_1(v)\nabla v \cdot \nabla v + \beta_2(v)\mathrm{tr}\, \mathbf{D}\}\mathbf{1} + \beta_3(v)\mathbf{D} + \beta_4(v)\nabla v \otimes \nabla v. \tag{3}$$

In the above equations v is the volume fraction of the solid, $\mathbf{D}$ is the symmetric part of the velocity gradient, $\beta_0(v)$ is similar to the pressure in a compressible fluid and is given by an equation of state, $\beta_2(v)$ is similar to the second coefficient of viscosity in a compressible fluid, $\beta_1(v)$ and $\beta_4(v)$ are the material parameters that reflect the distribution of the granular materials, and $\beta_3(v)$ is the viscosity of the granular materials. The above model allows for normal-stress differences, a feature observed in granular materials. In general, the material properties β_0 through β_4 are functions of the density (or volume fraction v), temperature, and the principal invariants of the appropriate kinematical quantities. The symmetric part of velocity gradient, $\mathbf{D}$, is given by

$$\mathbf{D} = \tfrac{1}{2}[(\nabla \mathbf{u}) + (\nabla \mathbf{u})^{\mathrm{T}}], \tag{4}$$

where $\mathbf{u}$ is the velocity of the particles. In Eq. (3), $\mathbf{1}$ is the identity tensor, ∇ is the gradient operator, $\otimes$ indicates the outer (dyadic) product of two vectors, and tr designates the trace of a tensor. Furthermore, v is related to the bulk density of the material ρ through

$$\rho = \rho_{\mathrm{s}} v, \tag{5}$$

where ρ_{s} is the actual density of the grains at position $\mathbf{x}$ and time t and the field v is called the volume fraction.

We now consider the flow of granular materials modeled by the above continuum model down an inclined plane due to the action of gravity. This problem has also been studied by Savage [10], Hutter et al. [32,33], Johnson and Jackson [34] and others, under the context of different models. In this problem, we assume steady one-dimensional flow of incompressible granular materials (i.e., $\rho_{\mathrm{s}} = \text{constant}$). Of course, ρ_{s} being constant does not imply that ρ is a constant since v can change.

In the present analysis we assume β_1, β_2 and β_4 to be constant and β_0 and the viscosity β_3 to be of the form [35,36].

$$\beta_0 = kv, \tag{6}$$

$$\beta_3 = \beta_{30}(v + v^2), \tag{7}$$

where β_{30} is a constant. For a justification of Eqs. (6) and (7) we refer the reader to Johnson et al. [35,36]. We should mention that in general β_1, β_2, and β_4 are also functions of v as discussed by Rajagopal and Massoudi [1]. The numerical simulations of Walton and Braun [37,38] suggest a quadratic variation in volume fraction. However, their analysis allows for the viscosity to vary with the shear rate, a feature that is not present in our work. Even so, at fixed shear rate, their simulation suggests a quadratic variation in the volume fraction, which is in keeping with our assumption.

For a fully developed flow of granular materials down an inclined plane, the density and the velocity fields are assumed to be of the form

$$v = v(y), \quad u = u(y), \tag{8}$$

The conservation of mass is automatically satisfied and the balance of linear momentum reduces to

$$k\frac{\mathrm{d}v}{\mathrm{d}y} + 2(\beta_1 + \beta_4)\frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}^2v}{\mathrm{d}y^2} = \rho_s g v \cos\alpha, \tag{9}$$

$$\beta_{30}(v + v^2)\frac{\mathrm{d}^2u}{\mathrm{d}y^2} + \beta_{30}(1 + 2v)\frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}u}{\mathrm{d}y} = -2\rho_s g v \sin\alpha, \tag{10}$$

where g denotes the acceleration due to gravity and α is the angle of inclination of the plane. We need to solve Eqs. (9) and (10) subject to the appropriate boundary conditions. We will discuss this in the next section.

3. Boundary conditions

A simple look at Eqs. (9) and (10) indicates that to have a well-posed problem, we need two boundary conditions for volume fraction, and two boundary conditions for velocity. This is unlike the fully developed flow of a linear incompressible fluid down an inclined plane where only two boundary conditions for the velocity are required. In general, whether we use the kinetic theory approach or the continuum approach, the need for additional boundary conditions arises. In the continuum theories of Goodman and Cowin [6,7] (where density gradient is included), and its modifications given by Ahmadi [11], Passman et al. [39], and Johnson et al. [35,36], two boundary conditions on the volume fraction are required. In the numerical solution of shearing motion of a fluid-solid flow, Passman et al. [39] prescribed the value of the volume fraction at the two plates. An alternative way is to use experimental results, if they are available. Later Johnson et al. [35,36] considered this issue and suggested using an integral condition for the volume fraction.

In the kinetic theory approach, additional boundary conditions are also necessary for the value of the fluctuating energy which is related to what is usually referred to as the granular temperature. There have been many attempts looking at these issues [32,33,40–44].

The effect of boundaries on the flow of granular materials has been studied experimentally by Savage and Sayed [44] and Hanes and Inman [45]. The same experiment was performed by the two groups, except that Savage and Sayed [44] roughened the walls of their shear cell with sand paper while Hanes and Inman [45] glued particles on the wall. Craig et al. [46] have also looked at the effect of boundary conditions.

Whether we use the continuum approach or the kinetic theory approach, slip may often occur at the wall, especially when the interstitial fluid is a gas, and therefore, the classical assumption of adherence boundary condition at the wall no longer applies. In our approach, we follow a similar procedure to that of Hutter et al. [32,33] in specifying the slip at the wall. Lugt and Schot [47] give a review of slip flow. While the phenomenon of slip at the wall occurs more frequently in the flow of rarefied gases and certain polymers, for the majority of fluid flows, the no-slip boundary condition is a reasonable one.

Perhaps, the idea of specifying slip at the wall goes back to Navier [48] who introduced a constant ς to describe slip at the wall

$$\varsigma u_s = \mu\frac{\mathrm{d}u}{\mathrm{d}n}, \tag{11}$$

where u_s is the slip velocity, μ is the fluid viscosity, u is the fluid velocity, and n is the normal of the wall directed into the fluid. Of course, there has been evidence for many years that for flows of some non-Newtonian fluids slip occurs at the wall. In fact, it is possible that the boundary condition is more complex in that the material 'stick-slips' on the boundary. If the shear stress is below a certain value, the material adheres to the boundary while it slips above a critical value of the shear stress. Such a phenomenon has been observed in polymeric materials and is the source of many surface instabilities observed in polymeric extrudates. An early attempt to generalize condition (11) for non-Newtonian fluids was made by Pearson and Petrie [49] in the form

$$u_s = f(\tau_w)\tau_w, \tag{12}$$

where τ_w is the wall shear stress. Hutter et al. [32,33] and Szidarovszky et al. [50] in their studies of granular materials used a similar relationship to relate the slip velocity, u_s, and the fluctuating energy of the flow, θ, of avalanches on an inclined plane. Specifically, they used

$$u_s = f(\tau^2)\tau, \tag{13}$$

$$\theta = g(u^2)u^2 \tag{14}$$

at $y = 0$, where

$$\tau = 2\chi\kappa(2 + \alpha)/5 \tag{15}$$

with

$$\kappa = \frac{1}{2}\frac{du}{dy}, \tag{16}$$

$$\chi = 2\rho^*\sigma(1 + e)\Phi_0(\theta/\pi)^{1/2}, \tag{17}$$

$$\Phi_0 = v^2 g_0 = \frac{v^2(2 - v)}{2(1 - v)^3}, \tag{18}$$

$$\rho = v\rho^*, \tag{19}$$

$$g_0 = \frac{1}{1 - v} + \frac{3v}{2(1 - v)^2} + \frac{v^2}{2(1 - v)^3}, \tag{20}$$

where σ is the particle diameter, e is the coefficient of restitution, and f and g depend on the surface roughness. Hutter et al. [32,33] suggested the following forms for f and g:

$$f(\chi^2) = C_f(\chi^2)^{(n-1)/2}, \tag{21}$$

$$g(\chi^2) = C_g(\chi^2)^{(m-1)/2}, \tag{22}$$

with sliding coefficient C_f and C_g and power-law exponents n and m. The case $C_f \to 0$ corresponds to the classical no-slip boundary condition. Szidarovszky et al. [50] provide an alternative boundary condition for (14) in the form of

$$a\theta + \chi\frac{d\theta}{dy} = 0 \tag{23}$$

which indicates that the gradient of the fluctuating energy at the wall is proportional to the fluctuating energy.

We will take a similar approach in that we assume that the slip velocity is proportional to the stress vector at the wall. That is

$$\mathbf{u}_s = \mathbf{f}[(\mathbf{Tn})_x, (\mathbf{Tn})_y], \tag{24}$$

where $\mathbf{T}$ is the stress tensor, $\mathbf{n}$ is a unit normal vector and $\mathbf{f}$ in general could be a function of surface roughness, volume fraction, shear rate, etc. With the constitutive equation (3), Eq. (24) becomes

$$u_s = f\left[\beta_0 \sin\alpha + \beta_1 \sin\alpha\left(\frac{dv_0}{dy}\right)^2 + \frac{\beta_3}{2}\cos\alpha\frac{du}{dy}\right], \tag{25}$$

where β_1 is assumed to be constant, as mentioned before, and β_0 and β_3 are given by Eqs. (6) and (7).

We need to solve Eqs. (9) and (10) subject to the appropriate boundary conditions. These boundary conditions for the velocity are:

at $y = 0$ (on inclined plane),

$$u = 0 \quad \text{(no-slip case)} \tag{26}$$

or

$$u = u_s \quad \text{(slip case)}, \tag{27}$$

where

$$u_s = f\left[kv\sin\alpha + \beta_1 \sin\alpha\left(\frac{dv_0}{dy}\right)^2 + \frac{\beta_{30}}{2}(v + v^2)\cos\alpha\frac{du}{dy}\right]. \tag{28}$$

The boundary conditions for the velocity and volume fraction at the free surface ($y = h$) are

$$kv + (\beta_1 + \beta_4)\left(\frac{dv}{dy}\right)^2 = 0, \quad \frac{du}{dy} = 0. \tag{29}$$

We still need one more condition for the volume fraction. One method is to use the following condition:

$$Q = \int_0^h v\,dy. \tag{30}$$

Note that we could, in theory, specify a value for v at $y = 0$ by either gluing particles to the wall, or by simply assuming a particle distribution at the wall. However, the condition given by Eq. (30), though strictly speaking is not a boundary condition, it nevertheless is a physical and reasonable condition to specify: it simply indicates an average value of the amount of particles in the system.

Also, notice that Eq. (29) is the stress-free condition. Johnson et al. [35,36] used Eq. (30) as an additional condition necessary for the specification of v. Furthermore, Rajagopal and Massoudi [1] and Rajagopal et al. [21] have shown that

$$k < 0 \tag{31}$$

as compression should lead to densification of the materials. In general, the second law of thermodynamics would also place restrictions on the nature of the material parameters; we will not discuss this issue here.

Now, the system of Eqs. (9) and (10) subject to the boundary conditions (26)–(30) are non-dimensionalized by

$$\xi = \frac{y}{h}, \quad u^* = \frac{u}{U_0}, \tag{32}$$

where h is a characteristic length and U_0 is a reference velocity. The above system of equations reduces to

$$R_1 \frac{\mathrm{d}v}{\mathrm{d}\xi} + R_2 \frac{\mathrm{d}v}{\mathrm{d}\xi}\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} = v\cos\alpha, \tag{33}$$

$$R_3 v(1+v)\frac{\mathrm{d}^2 u^*}{\mathrm{d}\xi^2} + R_3(1+2v)\frac{\mathrm{d}v}{\mathrm{d}\xi}\frac{\mathrm{d}u^*}{\mathrm{d}\xi} = -v\sin\alpha. \tag{34}$$

Eqs. (33) and (34) are subject to the following boundary conditions, in their dimensionless forms:

at $\xi = 1$ (at the free surface)

$$\frac{\mathrm{d}u^*}{\mathrm{d}\xi} = 0, \tag{35}$$

$$R_1 v + \frac{R_2}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}\xi}\right)^2 = 0. \tag{36}$$

The condition for the volume fraction v is

$$N = \int_0^1 v \,\mathrm{d}\xi. \tag{37}$$

Integrating Eq. (33) from $\xi = 0$ to 1, and using the boundary condition given by Eq. (36) and the volume fraction condition given by Eq. (37), the condition for the volume fraction at the surface is given by

$$\left[R_1 v + \frac{R_2}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}\xi}\right)^2\right]_{\xi=0} = -N\cos\alpha. \tag{38}$$

The dimensionless form of the boundary condition for velocity at $\xi = 0$ (on inclined plane) is

$$u^* = 0 \quad \text{(no-slip case)}, \tag{39}$$

$$u^* = u_s^* \quad \text{(slip case)}, \tag{40}$$

where

$$u_s^* = f^*\left[R_1 v\sin\alpha + R_{21}\sin\alpha\left(\frac{\mathrm{d}v_0}{\mathrm{d}\xi}\right)^2 + R_3(v+v^2)\cos\alpha\frac{\mathrm{d}u^*}{\mathrm{d}\xi}\right]. \tag{41}$$

Now, $f^* = hf/U$ and the non-dimensional parameters, R_1, R_2, R_3, and R_{21} are given by

$$R_1 = \frac{k}{h\rho_s g}, \quad R_2 = \frac{2(\beta_1 + \beta_4)}{h^3\rho_s g}, \tag{42}$$

$$R_3 = \frac{\beta_{30} U_0}{2h^2\rho_s g}, \quad R_{21} = \frac{\beta_1}{h^3\rho_s g}. \tag{43}$$

These dimensionless parameters have the following physical interpretations: parameter R_1 could be thought of as the ratio of the pressure force to the gravity force, R_2 is the ratio of the force due to volume fraction distribution to the gravity force, R_3 is the ratio of the viscous force to the gravity force (related to the Reynolds number). Since k is less than zero, R_1 can only have negative values, and since β_3 is positive, R_3 is only given positive values. R_{21} has a similar meaning to R_2.

4. Numerical calculations

Eqs. (33) and (34) subject to boundary conditions (35)–(41) are solved numerically to obtain the dimensionless velocity profiles, the surface velocities, the free stream velocities, and the velocity gradients at the surface using R_1, R_2, R_3, R_{21}, and f^* as parameters while N, and α are kept constant at 0.3 and 30°, respectively. To do so, Eqs. (33) and (34) are discretized using central-difference approximation. Since the conditions $du^*/d\xi$, and $dv/d\xi$ at $\xi = 0$ are not known, the numerical procedure requires that these unknowns must be initially guessed so that the calculation can proceed throughout the domain (from $\xi = 0$ to 1). The solutions of v and u^* at $\xi = 1$, provided by the initial guesses, are compared with the known values of v and u^* at $\xi = 1$, represented by Eqs. (35) and (36), respectively. If the solutions at this point are such that Eqs. (35) and (36) are not satisfied, another guess must be used and the calculation is repeated. This procedure is continued until the solutions of v and u^* at $\xi = 1$ provided by the initial guesses satisfy conditions (35) and (36). In order to reduce the guess work, the Newton–Raphson method is used for correcting the initial guesses. These procedures are described in detail by Phuoc and Massoudi [31].

5. Results and discussion

The solutions for the velocity profile, velocity gradient, surface velocity and the free stream velocity are presented for different values of the physical parameters R_1–R_3 and the slip factor. The effect of the slip factor is investigated using f^* as a variable while R_1, R_2, R_3, R_{21}, are kept constants. Typical results for this case are shown in Figs. 1 and 2. Fig. 1A shows a linear shift of the velocity profiles corresponding to four values of $f^* = 0$, 40, 80, and 120. Such a linear shift is seen clearly using information from Fig. 2A where the surface velocity and the free stream velocity are plotted against the slip factor showing that both the surface velocity and the free stream velocity increase linearly as the slip factor increases. The shape of these profiles, however, are the same for all four values of the slip factor. This can be seen in Figs. 1B and 2B where the velocity gradient for the entire flow and the velocity gradient at the surface of the inclined plane are seen to be independent of the slip factor. Thus, the results reported here show that increasing the slip factor does not alter the flow pattern. The only effect seems to be a speeding up of the flow leading to a linear increase in the flow velocity. A similar phenomenon was observed by Hutter et al. [33].

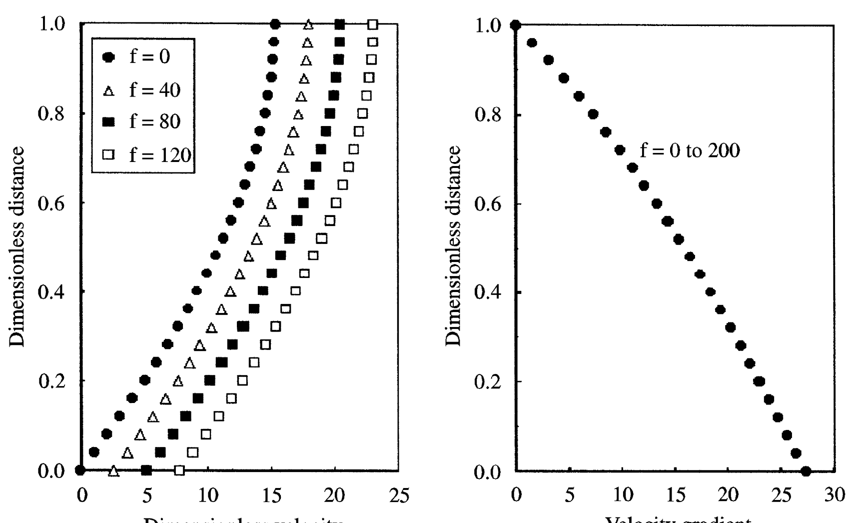

Fig. 1. Effects of the slip factor on the velocity and the velocity gradient ($R_1 = -1.0$, $R_2 = 10$, $R_3 = 0.01$, $R_{21} = 10$, $\alpha = 30$).

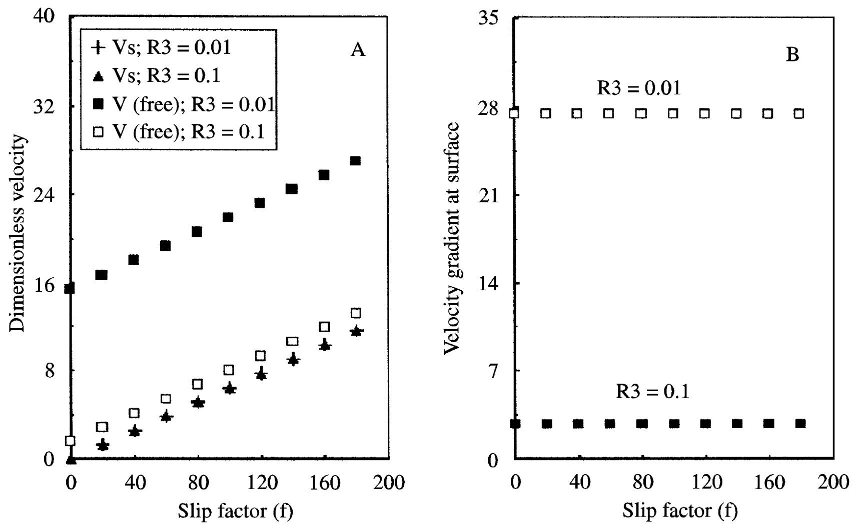

Fig. 2. Surface and free stream velocities and velocity gradient at the surface versus the slip factor: effects of R_3 ($R_1 = -1.0$, $R_2 = 10$, $R_{21} = 10$, $\alpha = 30$).

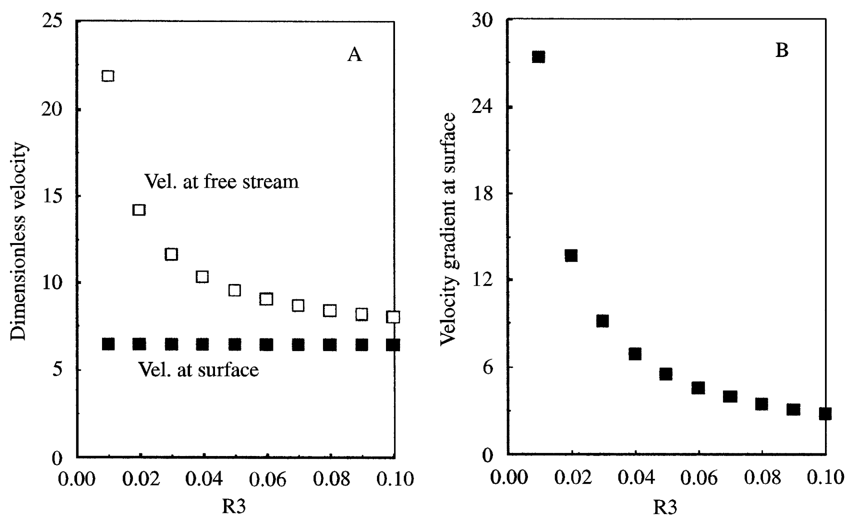

Fig. 3. Dimensionless velocity at surface and free stream and velocity gradient at the surface versus R_3 ($R_1 = -1.0$, $R_2 = 10$, $R_{21} = 10$, $\alpha = 30, f^* = 100$).

The effect of the parameter R_3 on the velocity at the surface and at the free surface, and the velocity gradient at the surface are shown in Figs. 3 and 4. As R_3 increases from 0.01 to 0.1, the surface velocity remains constant. On the other hand, the free stream velocity and the velocity gradient at the surface decrease drastically and become independent of R_3 when its value is higher than 0.09. Under this condition, the free stream velocity approaches the surface velocity flattening the velocity profile for the entire flow domain as shown in Fig. 4. Since R_3 is defined as the ratio of the

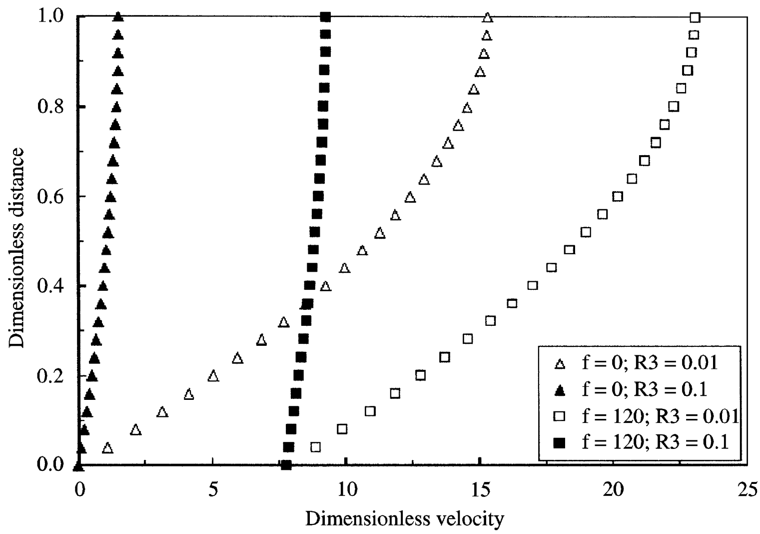

Fig. 4. Effects of the slip factor f^* and R_3 on the dimensionless velocity profiles ($R_1 = -1.0$, $R_2 = 10$, $R_{21} = 10$, $\alpha = 30$).

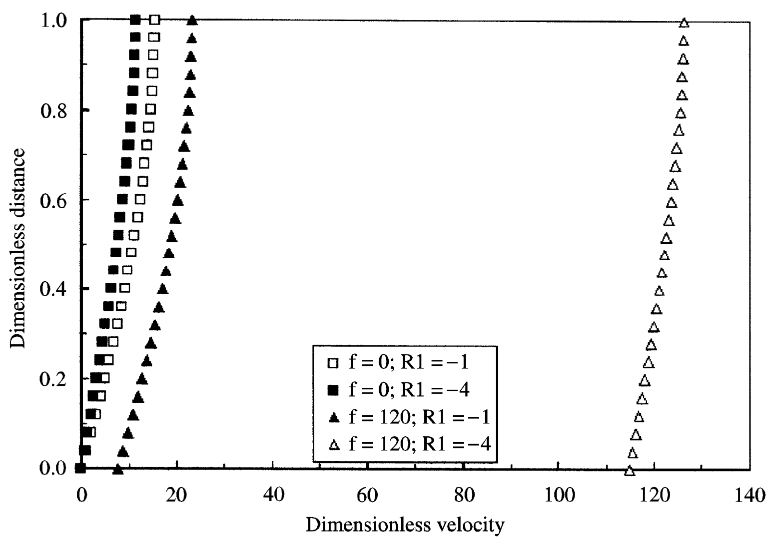

Fig. 5. Effects of the slip factor f^* and R_1 on the dimensionless velocity profiles ($R_3 = 0.01$, $R_2 = 10$, $R_{21} = 10$, $\alpha = 30$).

viscous force to the gravity force, this result indicates that, the effect of viscous force on the flow is very different from that due to gravity force. The viscous force tends to retard the free stream flow. Therefore, when the flow is under viscous control, the free stream velocity is close to the surface velocity and the velocity profile becomes flat. The gravity force, on the other hand, tends to increase the free stream velocity, therefore, if the flow is under gravity-control the free stream velocity is very much higher than the surface velocity.

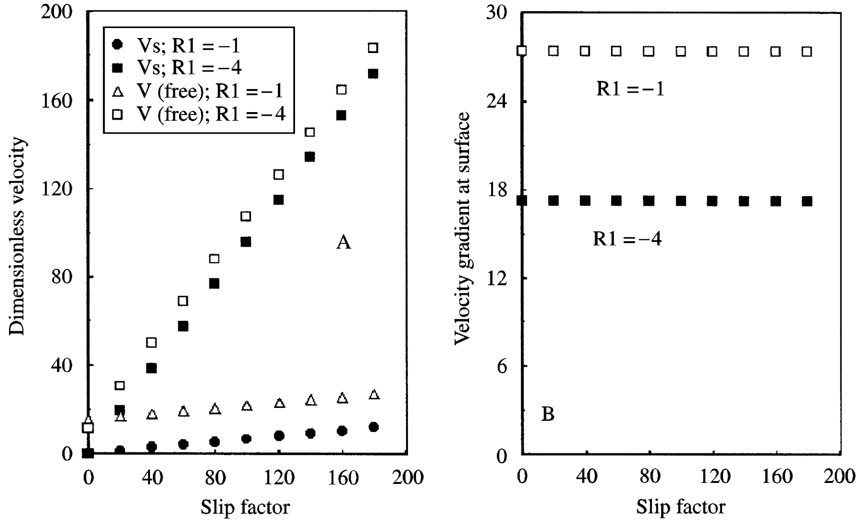

Fig. 6. Surface and free stream velocities and velocity gradient at the surface versus the slip factor: effects of R_1 ($R_3 = 0.01$, $R_2 = 10$, $R_{21} = 10$, $\alpha = 30$).

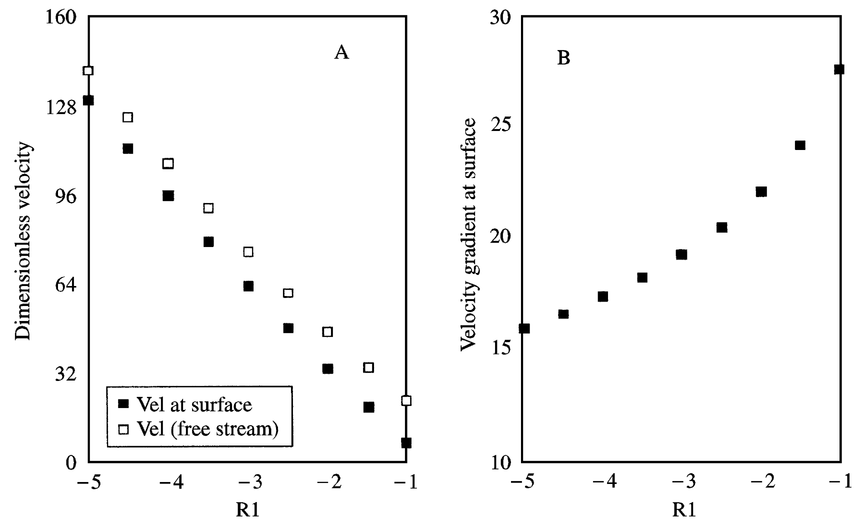

Fig. 7. Dimensionless velocity at surface and free stream and velocity gradient at the surface versus R_1 ($R_3 = 0.01$, $R_2 = 10$, $R_{21} = 10$, $\alpha = 30$, $f^* = 100$).

Since the velocity profiles of the flow field are determined by the velocity at the surface, it is clear from Eq. (41) that, when the slip factor f^* is equal to zero, R_1 has an insignificant effect on the flow field. When $f^* > 0$, however, the effect of R_1 becomes more significant. These results are shown in Figs. 5–7. Data in these figures are obtained with two values of R_1 ($R_1 = -1$ and -4), while other parameters were kept constant. Fig. 5 shows the slip/no-slip dimensionless velocity profiles. Fig. 6 shows the surface and free stream dimensionless velocity as a function of the slip factor, while in

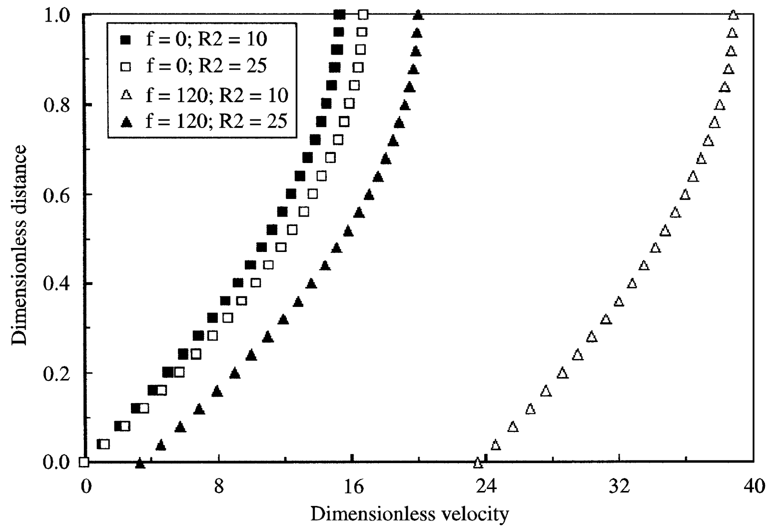

Fig. 8. Effects of the slip factor f^* and R_2 on the dimensionless velocity profiles (R_3 = 0.01, R_1 = − 1.0, R_{21} = 20, α = 30).

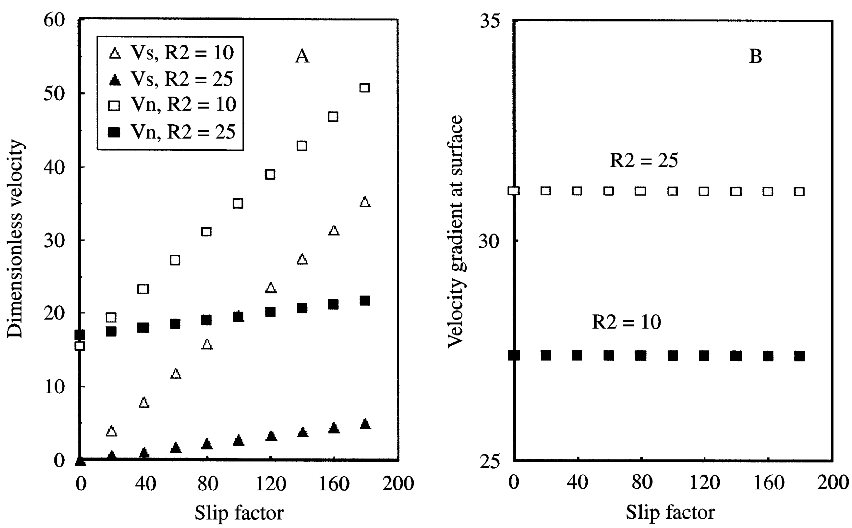

Fig. 9. Surface and free stream velocities and velocity gradient at the surface versus the slip factor: effects of R_2 (R_3 = 0.01, R_1 = − 1.0, R_{21} = 20, α = 30).

Fig. 7 we present the variation of these velocities with respect to R_1. The results indicate that, when $f^* = 0$, the velocity profile obtained for $R_1 = -1$ is very close to that obtained for $R_1 = -4$. However, when $f^* > 0$, these profiles are far apart from each other. Fig. 7 shows that when the absolute value of R_1 is increased the velocity gradient at the surface decreases and the surface as well as the free stream dimensionless velocities increase drastically.

Since R_2 is a measure of the volume distribution force to the gravity force, we can observe the effect of the volume distribution force on the flow field by

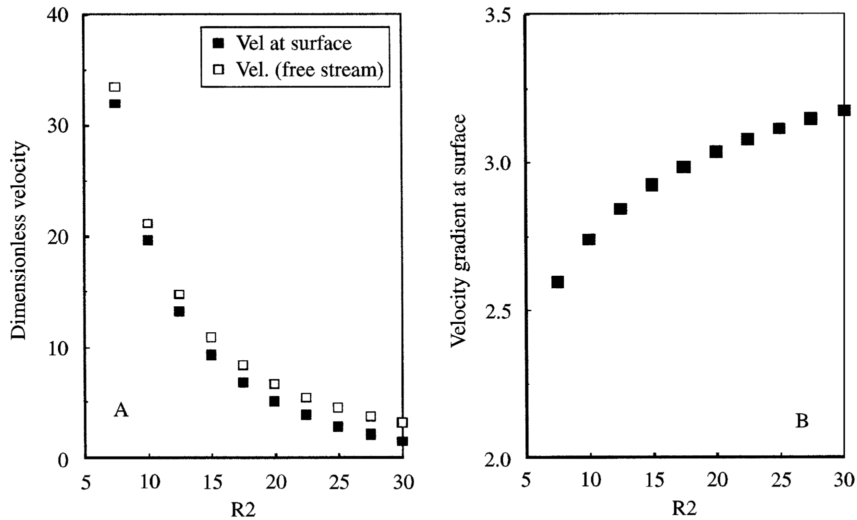

Fig. 10. Dimensionless velocity at surface and free stream and velocity gradient at the surface versus R_2 ($R_3 = 0.1$, $R_1 = -1.0$, $R_{21} = 20$, $\alpha = 30$, $f^* = 100$).

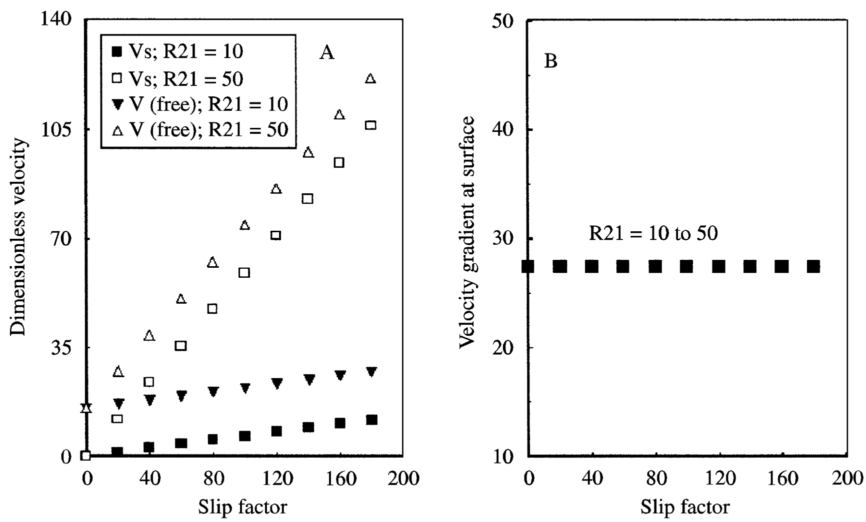

Fig. 11. Surface and free stream velocities and velocity gradient at the surface versus the slip factor: effects of R_{21} ($R_3 = 0.01$, $R_1 = -1.0$, $R_2 = 10$, $\alpha = 30$).

changing R_2 while keeping other parameters constant. Some typical results are presented in Figs. 8–10. Similar to R_1, the effect of parameter R_2 is observed to be significant only when the slip factor $f^* > 0$. As seen from Fig. 10, while R_2 has a strong effect on the surface and free stream dimensionless velocities, it has only a minor effect on the velocity gradient at the surface. For example, when R_2 increases from 5 to 30 the surface and free stream dimensionless velocities decrease from about 35 to less than 5 while the velocity gradient at the surface slightly increases from about 2.6 to

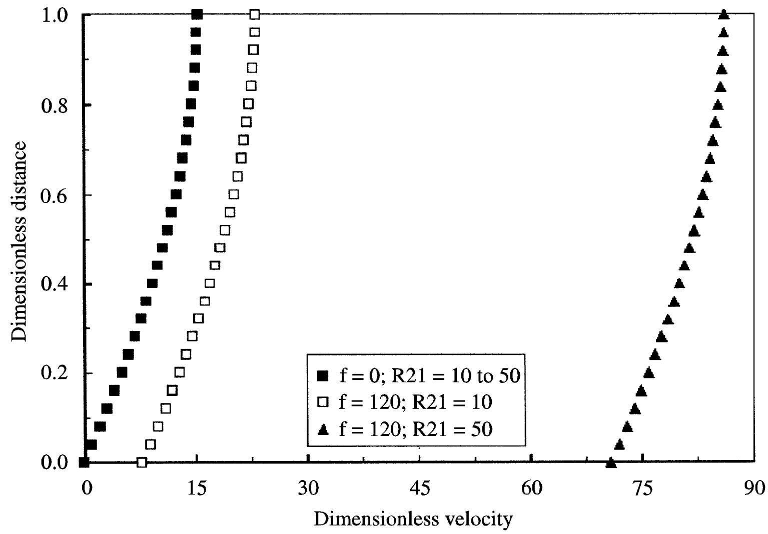

Fig. 12. Effects of the slip factor f^* and R_{21} on the dimensionless velocity profiles ($R_3 = 0.01$, $R_1 = -1.0$, $R_2 = 10$, $\alpha = 30$).

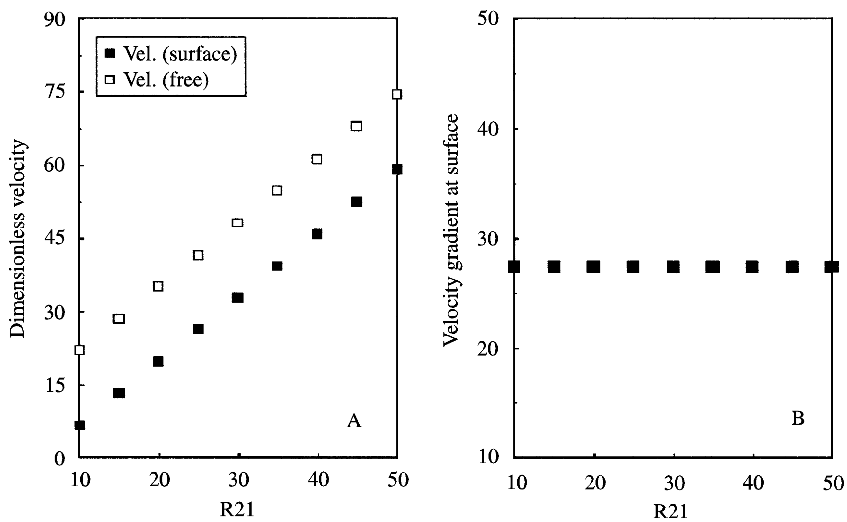

Fig. 13. Dimensionless velocity at surface and free stream and velocity gradient at the surface versus R_{21} ($R_3 = 0.01$, $R_1 = -1.0$, $R_2 = 20$, $\alpha = 30$, $f^* = 100$).

3.5 (see Fig. 10). Since the shape of the velocity profile depends on the velocity gradient at the surface, such a small increase in the velocity gradient at the surface does not have a significant effect on the shape of the velocity profile as shown in Fig. 8. Thus, it can be said that the volume distribution force significantly affects the flow field when the slip factor $f^* > 0$. In this case, an increase in the volume distribution force leads to a significant decrease in the flow field velocity but the shape of the velocity profile remains unchanged.

Figs. 11–13 show the effects of the parameter R_{21} on the flow properties. Since R_{21} has the same physical meaning as R_2, the effect of R_{21} on these flow properties should be similar to that of R_2. In Fig. 12, when $f^* = 0$, the velocity profiles remain unchanged for R_{21} varying from 10 to 50. For $f^* = 120$, although the shape of the velocity profile remains undisturbed, an increase in R_{21} leads to a linear shift in the velocity profiles. The increase in the magnitude of the surface and free stream dimensionless velocities and the velocity gradient at the surface as a function of R_{21} is presented in Fig. 13. It is clear that, when R_{21} is increased from 10 to 50 the velocity gradient at the surface remains constant while both the surface and free stream velocities increase linearly.

6. Conclusions

The dense flow of granular materials down an inclined plane, using the constitutive relation used by Rajagopal and Massoudi subject to slip at the wall is studied. The flow is assumed to be steady and fully developed, and as a result, phenomenon such as 'hydraulic jump' observed by Brennen and co-workers and Jackson and co-workers is not allowed in the present case. Due to the kinematical constraints, the equation for the volume fraction can be solved independent of the momentum equation. As a result, the slip boundary condition only affects the velocity profiles. At the free surface, the stress-free condition is imposed. The parametric study, while varying the dimensionless numbers, indicates that when the slip factor is zero, we recover the results obtained by Gudhe et al. [23], and when the slip factor is not zero, a range of interesting phenomenon is observed for different values of the dimensionless numbers.

We need to mention here that even though we have referred to the works of Hutter et al. [32,33] extensively, we are not able to compare the results of our investigation with theirs in a quantitative manner. The main reason is that though Hutter et al. have studied the same problem with similar slip boundary condition at the wall, since they have taken a kinetic theory approach, they have additional parameters such as the granular temperature, which is really a measure of the fluctuation of the particles, and not a real temperature measured by a thermometer. As a result, they also have an additional governing equation which resembles the heat transfer equation. Having said that, we do observe, in a qualitative way, similar velocity profiles to those of Hutter et al.

Finally, a look at Eq. (9) or (33) would reveal that the volume fraction distribution, i.e., the density field, for this particular flow is independent of the velocity field. As a result, the imposition of the slip boundary condition has no effect on the volume fraction distribution. This is purely due to (i) the kinematical assumption given by Eq. (8), and (ii) the fact that we have ignored the effect of the interstitial fluid. That is, if we now consider the flow of a granular material modeled by Eq. (3) in a chute, the flow field is at least two-dimensional, and in this case the velocity distribution will have an effect on ν. In the second case, where we have to use the Mixture Theory (or take a multi-component approach), even for a one-dimensional fully developed flowfield of the type given in Eq. (8), due to the interactive forces such as lift, or pressure of the fluid, there would be lateral movement of the particles, and as a result a change in the boundary condition for the flow at the solid surface will have an effect on the volume fraction.

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